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LETTER TO THE EDITOR

On the integrability of modified Lax equations

B A Kupershmidt

University of Tennessee Space Institute, Tullahoma, TN 37388, USA

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Abstract. The modified Lax equations resulting from factorisation of scalar Lax operators are shown to commute between themselves. The proof is general and bypasses the Hamiltonian formalism of Lax equations. In particular, the modified KP hierarchy and modified generalised Toda lattices are proved to be integrable.

The purpose of this letter is to show that when one factorises a scalar Lax operator, the resulting modified flows commute between themselves even when there is no information available on the Hamiltonian structure of the modified equations. Such a non-Hamiltonian situation occurs, e.g., when one factorises the Lax operator of the KP hierarchy [1]:

$$L = \xi + \sum_{i=0}^{\infty} u_i \xi^{-i-1}. \tag{1}$$

Here

$$\xi = \partial = \partial/\partial x \tag{2}$$

and the mth flow of the KP hierarchy has the form

$$L_t = [(L^m)_+, L] = [-(L^m)_-, L] \qquad m \in \mathbb{N} \tag{3}$$

where

$$\left(\sum p_k \xi^k\right)_+ := \sum_{k \geq 0} p_k \xi^k \qquad \left(\sum p_k \xi^k\right)_- := \sum_{k<0} p_k \xi^k. \tag{4}$$

In the KP case, the factorisation ansatz is [2]

$$L = l_1 l_2 \qquad \text{or} \qquad L = l_2 l_1 \tag{5}$$

with

$$l_1 = \xi - v_0 \qquad l_2 = 1 + \sum_{i=0}^{\infty} v_i \xi^{-i-1} \tag{6}$$

and the modified motion equations are of the form

$$l_{1,t} = ((l_1 l_2)^m)_+ l_1 - l_1((l_2 l_1)^m)_+ = l_1((l_2 l_1)^m)_- - ((l_1 l_2)^m)_- l_1 \tag{7a}$$

$$l_{2,t} = ((l_2 l_1)^m)_+ l_2 - l_2((l_1 l_2)^m)_+ = l_2((l_1 l_2)^m)_- - ((l_2 l_1)^m)_- l_2. \tag{7b}$$

These equations are called 'modified' because they imply the equations:

$$(l_1 l_2)_t = [((l_1 l_2)^m)_+, l_1 l_2] = [l_1 l_2, ((l_1 l_2)^m)_-] \tag{8a}$$

$$(l_2 l_1)_t = [((l_2 l_1)^m)_+, l_2 l_1] = [l_2 l_1, ((l_2 l_1)^m)_-]. \tag{8b}$$

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In the first non-trivial case $m=2$, the equations (7) become [2]

$$\tfrac{1}{2}v_{i,t}=\tfrac{1}{2}\partial^2(v_i)+\partial(v_{i+1})+v_i(v_1-v_0^2)-\sum_{j=0}^{i}(-1)^j\binom{i}{j}v_{i-j}\partial^j(v_1-v_0^2+\partial(v_0)). \tag{9}$$

The Hamiltonian structure of the modified KP systems (7) is not known, and its determination is far beyond the technical capabilities of the modern Hamiltonian formalism; the same conclusion applies to the system (9) and even to its quasiclassical (i.e. zero-dispersion) limit:

$$\tfrac{1}{2}v_{i,t}=\partial(v_{i+1})-v_i\partial(v_0)+\mathrm{i}v_{i-1}\partial(v_1-v_0^2). \tag{10}$$

If we know both the Hamiltonian structure of the modified equations and the property of the factorisation map to be canonical, we can conclude at once that the modified flows commute between themselves since their respective Hamiltonians are in involution, being pull-backs of the involutive Hamiltonians in the unmodified space. Such knowledge is sometimes available [3], but in general it is not [4-6], and it is plainly desirable to have a general proof of commutativity of modified flows. Such a proof is offered below.

Let

$$L=\xi^n+\sum_{s=q}^{n-2}u_s\xi^s \qquad n\in\mathbb{N} \tag{11}$$

be a scalar Lax operator, with

$$q=0 \qquad \text{or} \qquad q=-\infty \tag{12}$$

let

$$P=L^{m/n}=\xi^m+\dots \tag{13}$$

be a generator of the centraliser $Z(L)$ of L, and let [7]

$$L_t=\partial_P(L)=[P_+,L]=[L,P_-] \tag{14}$$

be the corresponding Lax equations. Set

$$l_r=\xi^{n(r)}+\sum_{s=q}^{n(r)-1}v_{rls}\xi^s \qquad r\in\{1,\dots,N\}=\mathbb{Z}_N \tag{15}$$

with

$$\sum_{r=1}^{N}n(r)=n \tag{16}$$

$$\sum_{r=1}^{N}v_{rln(r)-1}=0. \tag{17}$$

Set

$$\underset{r}{L}:=l_rl_{r+1}\dots l_{r-1} \qquad r\in\mathbb{Z}_N. \tag{18}$$

Equation (18) defines a ('Miura map') homomorphism $\phi_r:\{u\}\to\{v\}$.

Set

$$\underset{r}{P}:=\phi_r(P)=(\underset{r}{L})^{m/n} \tag{19}$$

and define the modified Lax equations by the formulae

$$\partial_P(l_r) = \underset{r}{P_+} l_r - l_r \underset{r+1}{P_+} \tag{20a}$$

$$= l_r \underset{r+1}{P_-} - \underset{r}{P_-} l_r \qquad r \in \mathbb{Z}_N. \tag{20b}$$

The second equality (20*b*) follows from the first since

$$\underset{r}{P} l_r = l_r \underset{r+1}{P} \tag{21}$$

which can be seen as follows:

$$\begin{aligned}
(l_r^{-1} \underset{r}{P} l_r)^n &= [l_r^{-1} (\underset{r}{L})^{m/n} l_r]^n \qquad \text{by equation (19)} \\
&= l_r^{-1} (\underset{r}{L})^m l_r \\
&= l_r^{-1} (l_r \ldots l_{r-1}) \ldots (l_r \ldots l_{r-1}) \qquad \text{by equation (18)} \\
&= (l_{r+1} \ldots l_r) \ldots (l_{r+1} \ldots l_r) = (l_{r+1} \ldots l_r)^m \\
&= (\underset{r+1}{L})^m \qquad \text{by equation (18)} \\
&= [(\underset{r+1}{L})^{m/n}]^n \\
&= (\underset{r+1}{P})^n \qquad \text{by equation (19)}
\end{aligned}$$

and both $l_r^{-1} \underset{r}{P} l_r$ and $\underset{r+1}{P}$ have the same leading term ξ^m.

Now, (20*b*) shows that

$$\operatorname{ord}[\partial_P(l_r)] \leqslant n(r) - 1$$

where

$$\operatorname{ord}\left(\sum p_k \xi^k\right) := \max\{k | p_k \neq 0\}.$$

In addition, when $q = 0$, (20*a*) shows that

$$\partial_P(l_r) = [\partial_P(l_r)]_+.$$

Finally, (20*b*) shows that

$$\partial_P(v_{rln(r)-1}) = \operatorname{Res}(\underset{r+1}{P} - \underset{r}{P}) \tag{22}$$

where

$$\operatorname{Res}\left(\sum p_k \xi^k\right) := p_{-1}.$$

Hence,

$$\partial_P\left(\sum_r v_{rln(r)-1}\right) = 0 \tag{23}$$

so that the constraint (17) is preserved by the dynamics. Thus, formulae (20) provide well defined equations in the space of modified variables. These are truly modified equations since they imply that

$$\partial_P(\underset{r}{L}) = [\underset{r}{P_+}, \underset{r}{L}] = [\underset{r}{L}, \underset{r}{P_-}]. \tag{24}$$

Indeed,

$$\begin{aligned}
\partial_P(\underset{r}{L}) &= \partial_P(l_r l_{r+1} \ldots l_{r-1}) \qquad \text{by equation (18)}\\
&= (\underset{r}{P_+} l_r - l_r \underset{r+1}{P_+}) l_{r+1} \ldots l_{r-1} + l_r(\underset{r+1}{P_+} l_{r+1} - l_{r+1} \underset{r+2}{P_+}) l_{r+2} \ldots l_{r-1} + \ldots\\
&\quad + l_r \ldots l_{r-2}(\underset{r-1}{P_+} l_{r-1} - l_{r-1} \underset{r}{P_+}) \qquad \text{by equation (20}a\text{)}\\
&= \underset{r}{P_+} l_r \ldots l_{r-1} - l_r \ldots l_{r-1} \underset{r}{P_+}\\
&= [\underset{r}{P_+}, \underset{r}{L}] \qquad \text{by equation (18).}
\end{aligned}$$

Now we can show that the modified flows commute between themselves. Let

$$Q = L^{\bar{m}/n} \qquad \bar{m} \in \mathbb{N} \tag{25}$$

be another generator of the centraliser $Z(L)$. Then, by (20),

$$\partial_Q(l_r) = \underset{r}{Q_+} l_r - l_r \underset{r+1}{Q_+} \tag{26}$$

hence, by (24),

$$\partial_Q(\underset{r}{L}) = [\underset{r}{L}, \underset{r}{Q_-}] \tag{27}$$

thus, by (19),

$$\partial_Q(\underset{r}{P}) = [\underset{r}{P}, \underset{r}{Q_-}] \tag{28}$$

so that

$$\partial_Q(\underset{r}{P_+}) = [\partial_Q(\underset{r}{P})]_+ = [\underset{r}{P}, \underset{r}{Q_-}]_+ = [\underset{r}{P_+}, \underset{r}{Q_-}]_+ \tag{29}$$

the last equality following from the obvious identity:

$$[(\ )_-, (\ )_-]_+ = 0. \tag{30}$$

We want to prove that

$$[\partial_Q, \partial_P](l_r) = 0 \qquad r \in \mathbb{Z}_N. \tag{31}$$

Temporarily denoting l_r by l, we get

$$\begin{aligned}
\partial_Q \partial_P(l) &= \partial_Q(\underset{r}{P_+} l - l \underset{r+1}{P_+}) \qquad \text{by equation (20}a\text{)}\\
&= [\underset{r}{P_+}, \underset{r}{Q_-}] l + \underset{r}{P_+}(\underset{r}{Q_+} l - l \underset{r+1}{Q_+}) - (\underset{r}{Q_+} l - l \underset{r+1}{Q_+}) \underset{r+1}{P_+} - l[\underset{r+1}{P_+}, \underset{r+1}{Q_-}]_+\\
&\quad \text{by equations (26), (29).}
\end{aligned} \tag{32}$$

Interchanging Q and P in (32), we obtain

$$\partial_P \partial_Q(l) = [\underset{r}{Q_+}, \underset{r}{P_-}]_+ l + \underset{r}{Q_+}(\underset{r}{P_+} l - l \underset{r+1}{P_+}) - (\underset{r}{P_+} l - l \underset{r+1}{P_+}) \underset{r+1}{Q_+} - l[\underset{r+1}{Q_+}, \underset{r+1}{P_-}]_+. \tag{33}$$

Hence, subtracting formula (33) from formula (32) we get

$$[\partial_Q, \partial_P](l) = A_r l - l A_{r+1} \tag{34}$$

where

$$A_r := [\underset{r}{P_+}, \underset{r}{Q_-}]_+ + [\underset{r}{P_-}, \underset{r}{Q_+}]_+ + [\underset{r}{P_+}, \underset{r}{Q_+}]. \tag{35}$$

But, by (30),

$$\begin{aligned} A_r &= [\underset{r}{P_+} + \underset{r}{P_-}, \underset{r}{Q_+} + \underset{r}{Q_-}]_+ = [\underset{r}{P}, \underset{r}{Q}]_+ \\ &= [\underset{r}{(L)}^{m/n}, \underset{r}{(L)}^{\bar{m}/n}]_+ \qquad \text{by equations (19), (25)} \\ &= 0 \end{aligned}$$

so that formula (34) becomes the desired formula (31).

Remark 1. The equations (20) can be written in the *matrix* Lax form [5, 8]:

$$\partial_P(\bar{L}) = [\bar{P}_+, \bar{L}] = [\bar{L}, \bar{P}_-] \tag{36}$$

where

$$\bar{L} := \begin{pmatrix} 0 & l_1 & & & \\ & 0 & l_2 & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & 0 & l_{N-1} \\ l_N & & & & 0 \end{pmatrix} \tag{37}$$

$$\bar{P} := \begin{pmatrix} \underset{1}{P} & & & & \\ & \underset{2}{P} & \mathbf{0} & & \\ & & \ddots & & \\ & \mathbf{0} & & \underset{N}{P} & \\ & & & & \underset{N-1}{P} \end{pmatrix}. \tag{38}$$

However, we cannot use the theory of *matrix* Lax equations [7] to prove the commutativity of the modified flows, since the matrix operator $\bar{L}$ (37) does not belong to the general class of regular semisimple matrix differential operators.

Remark 2. From the proof of formula (31) it follows that the modified KP hierarchy (7) is commutative; this fact has not been known previously.

Remark 3. Exactly the *same* proof as above applies to modified discrete Lax equations from [8, chapter 4]. Thus, all discrete modified Lax equations, whether Hamiltonian or not, are integrable.

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